UNIVERSAL ABELIAN GROUPS

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ABSTRACT

We examine the existence of universal elements in classes of infinite abelian groups. The main method is using group invariants which are defined relative to club guessing sequences. We prove, for example:

THEOREM: *For $n \geq 2$, there is a purely universal separable p-group in $\aleph_n$ if, and only if, $2^{\aleph_0} \leq \aleph_n$.*

0. Introduction

In this paper "group" will always mean "infinite abelian group", and "cardinal" and "cardinality" always refer to infinite cardinals and infinite cardinalities.

Given a class of groups K and a cardinal λ we call a group $G \in K$ **universal for K in λ** if $|G| = \lambda$ and every $H \in K$ with $|H| \leq \lambda$ is isomorphic to a subgroup of G. The objective of this paper is to examine the existence of universal groups in various well-known classes of infinite abelian groups. We also investigate the existence of **purely universal** groups for K in λ, namely groups $G \in K$ with $|G| = \lambda$ such that every $H \in K$ with $|H| \leq \lambda$ is isomorphic to a **pure** subgroup of G.

The main set theoretic tool we use is a *club guessing sequence*. This is a prediction principle which has enough power to control properties of an infinite object

which are defined by looking at all possible enumerations of the object. Unlike the diamond and the square, two combinatorial principles which are already accepted as useful for the theory of infinite abelian groups, club guessing sequences are proved to exist in ZFC. Therefore using club guessing sequences does not require any additional axioms beyond the usual axioms of ZFC. Club guessing sequences are particularly useful in proving theorems from **negations** of CH and GCH.

The paper is organized as follows: in Section 1 we define group invariants relative to club guessing sequences, and show that the invariants are monotone in pure embeddings. In Section 2 we construct various groups with prescribed demands on their invariants. In Section 3 these ideas are used to investigate the existence of universal groups for classes of torsion groups and classes of torsion free groups. It appears that cardinal arithmetic decides the question of existence of a universal group in many cardinals. For example: there is a purely universal separable p-group in $\aleph_n$ iff $\aleph_n \geq 2^{\aleph_0}$ for all $n \geq 2$ (for $n = 1$ only the "if" part holds).

This paper follows two other papers by the same authors, [KjSh 409] and [KjSh 447], in which the existence of universal linear orders, boolean algebras, and models of unstable and stable unsuperstable first order theories were examined using the same method.

All the abelian group theory one needs here, and more, is found in [Fu], whose system of notation we adopt. An acquaintance with ordinals and cardinals is necessary, as well as familiarity with stationary sets and the closed unbounded filter. Knowledge of chapter II in [EM] is more than enough.

Before proceeding, we first observe that in every infinite cardinality there are universal groups which are divisible:

0.1 THEOREM: *In every cardinality there is a universal group, universal p-group (for every prime p), universal torsion group and universal torsion-free group.*

Proof: These are, respectively, the direct sum of λ copies of the rational group Q together with λ copies of $Z(p^\infty)$ for every prime p; the direct sum of λ copies of $Z(p^\infty)$; the direct sum of λ copies of $Z(p^\infty)$ for every prime p; and the direct sum of λ copies of Q. The universality of these groups for their respective classes follows from the structure theorem for divisible groups and the fact that every group (p-group, torsion group, torsion-free group) is embeddable in a divisible

group (p-group, torsion group, torsion-free group) of the same cardinality ([Fu] I, 23 and 24]).   ∎

1. The invariant of a group relative to the ideal id($\overline{C}$)

A fixed assumption in this section is that λ is a regular uncountable cardinal. We assume the reader is familiar with the basic properties of closed unbounded sets of λ, and with the definition and basic properties of stationary sets.

1.1 Definition: For a group G, $nG \stackrel{\text{def}}{=} \{ng: g \in G\}$. Two elements $g, h \in G$ are *n*-**congruent** if $g - h \in nG$. If g, h are *n*-congruent, we also say that h is an *n*-**congruent** of g.

1.2 Definition:
  (1) ([Fuchs, p.113]) Let G be group. A subgroup $H \subseteq G$ is a **pure** subgroup, denoted by $H \subseteq_{pr} G$, if for all natural n, $nH = nG \cap H$.
  (2) An embedding of groups $h: H \to G$ is a **pure** embedding if its image $h(H)$ is a pure subgroup of G.

1.3 Definition: Suppose that λ is a regular uncountable cardinal and that G is a group of cardinality λ. A sequence $\overline{G} = \langle G_\alpha: \alpha < \lambda \rangle$ is called a λ-**filtration** of G iff for all α
  (1) $G_\alpha \subseteq G_{\alpha+1}$,
  (2) G_α is of cardinality smaller than λ,
  (3) if α is limit, then $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$,
  (4) $G = \bigcup_{\alpha < \lambda} G_\alpha$.
  Suppose $\overline{G} = \langle G_\alpha: \alpha < \lambda \rangle$ is a given filtration of a group G. Suppose $c \subseteq \lambda$ is a set of ordinals, and the increasing enumeration of c is $\langle \alpha_i: i < i(*) \rangle$. Let $g \in G$ be an element.

1.4 Definition: $\text{Inv}_{\overline{G}}(g, c) = \{\alpha_i \in c: g \in \bigcup_n ((G_{\alpha_i+1} + nG) - (G_{\alpha_i} + nG))\}$.
  We call $\text{Inv}_{\overline{G}}(g, c)$ **the invariant of the element g relative to the λ-filtration $\overline{G}$ and the set of indices c.**

  Worded otherwise, $\text{Inv}_{\overline{G}}(g, c)$ is the subset of those indices α_i such that, by increasing the group G_{α_i} to the larger group $G_{\alpha_{i+1}}$, an n-congruent for g is introduced for some n.

  As the definition of the invariant depends on a λ-filtration, one may think that the invariant does not deserve its name. Indeed, given a group G equipped with

two respective λ-filtrations $\overline{G}$ and $\overline{G}'$, it is not necessarily true that for $g \in G$

(1) $$\operatorname{Inv}_{\overline{G}}(g,c) = \operatorname{Inv}_{\overline{G}'}(g,c).$$

The solution to this problem is working with a club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ and the ideal $\operatorname{id}(\overline{C})$ associated to it. The idea is as follows: for any pair of λ-filtrations $\overline{G}$ and $\overline{G}'$ a group G there is a club $E \subseteq \lambda$ such that for every $\alpha \in E$, $G_\alpha = G'_\alpha$. So if we chose our set c in the definition of invariant to consist only of such "good" α-s, namely if $c \subseteq E$, then it does not matter according to which λ-filtration we work. But we cannot choose a set c which is a subset of every club E resulting from some pair of λ-filtrations. What we **can** do is find a sequence of c-s with the property that for every club $E \subseteq \lambda$, stationarily many of them are subsets of E. Thus we will be able to define an invariant that is independent of a particular choice of a λ-filtration. Here is the precise formulation of this:

1.5 Definition: A sequence $\langle c_\delta : \delta \in S \rangle$, where $S \subseteq \lambda$ is a stationary set, $c_\delta \subseteq \delta$ and $\delta = \sup c_\delta$ for every δ, is called a **club guessing sequence** if for every club $E \subseteq \lambda$ the set $\{\delta \in S : c_\delta \subseteq E\}$ is a stationary subset of λ.

The theorems asserting the existence of club guessing sequences will be quoted later. A club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ gives rise to an ideal $\operatorname{id}(\overline{C})$ over λ — the **guessing ideal**.

1.6 Definition: Suppose that $\overline{C} = \langle c_\delta : \delta \in S \rangle$ is a club gussing sequence. We define a proper ideal $\operatorname{id}(\overline{C})$ as follows:

$A \in \operatorname{id}(\overline{C}) \iff A \subseteq \lambda \ \& \ \exists E \subseteq \lambda, \ E \text{ club}, \ \& \ \forall \delta \in E \cap S, c_\delta \not\subseteq E.$

So a set of δ-s is small if there is a club E which it fails to guess stationarily often, namely there is no $\delta \in A$ such that $c_\delta \subseteq E$.

1.7 LEMMA: If $\overline{C}$ is a club guessing sequence as above, then $\operatorname{id}(\overline{C})$ is a proper, λ complete ideal.

Proof: That $\operatorname{id}(\overline{C})$ is proper means that it does not contains every subset of λ. Indeed, $S \notin \operatorname{id}(\overline{C})$, as it guesses every club. That $\operatorname{id}(\overline{C})$ is downward closed is immediate from the definition. Suppose, finally, that $A_i, i < (*)$, $i(*) < \lambda$ are sets in the ideal. We show that their union $A \overset{\text{def}}{=} \bigcup_{i<i(*)} A_i$ is in the ideal. Pick a club E_i for every $i < i(*)$ so that $\delta \in A_i \Rightarrow c_\delta \not\subseteq E_i$. The set $E = \bigcap_{i<i(*)} E_i$ is a club. Suppose that $d \in A$. Then there here is some $i < i(*)$ such that $\delta \in A_i$. Therefore $c_\delta \not\subseteq E_i$. But $E \subseteq E_i$, so necessarily $c_\delta \not\subseteq E$. Thus, $A \in \operatorname{id}(\overline{C})$.   ∎

We adopt the phrase "for almost every δ in S", by which we mean "for all $\delta \in S$ except for a set in $\mathrm{id}(\overline{C})$".

1.8 LEMMA: *Suppose that $\overline{C} = \langle c_\delta : \delta \in S \rangle$ is a club guessing sequence on $S \subseteq \lambda$. Suppose that $\overline{G}$ and $\overline{G}'$ are two λ-filtrations of a group G of cardinality λ. Then for almost every $\delta \in S$, (1) holds for every $g \in G$.*

Proof: The set of $\alpha < \lambda$ for which $G_\alpha = G'_\alpha$ is a club. Let us denote it by E. If for some δ, $c_\delta \subseteq E$ holds, then for every $g \in G$ it is true that $\mathrm{Inv}_{\overline{G}}(g, c_\delta) = \mathrm{Inv}_{\overline{G}'}(g, c_\delta)$. But as $\overline{C}$ is a club guessing sequence, by definition, for almost every δ, $c_\delta \subseteq E$. ∎

We define now the desired group invariant.

1.9 Definition: Suppose that $\overline{C}$ is a club guessing sequence and that $\overline{G}$ is a λ-filtration of a group G of cardinality λ. Let

(1) $P_\delta(\overline{G}, \overline{C}) = \{\mathrm{Inv}_{\overline{G}}(g, c_\delta) : g \in G\}$,

(2) $\mathrm{Inv}(G, \overline{C}) = [\langle P_\delta(\overline{G}, \overline{C}) : \delta \in S \rangle]_{\mathrm{id}(\overline{C})}$.

The second item should read "the equivalence class of the sequence of P_δ modulo the ideal $\mathrm{id}(\overline{C})$", where two sequences are equivalent modulo an ideal if the set of coordinates in which the sequences differ is in the ideal.

1.10 LEMMA: *The definition of $\mathrm{Inv}(G, \overline{G})$ does not depend on the choice λ-filtration.*

Proof: Suppose that $\overline{G}, \overline{G}'$ are two λ-filtrations. By the regularity of λ, there exists a club E such that for every $\alpha \in E$, $G_\alpha = G'_\alpha$. Therefore for every δ such that $c_\delta \subseteq E$ and every $g \in G$, $\mathrm{Inv}_{\overline{G}}(g, c_\delta) = \mathrm{Inv}_{\overline{G}'}(g, c_\delta)$. This means that for every δ such that $c_\delta \subseteq E$, $P_\delta(\overline{G}, \overline{C}) = P_\delta(\overline{G}', \overline{C})$. But for almost all δ it is true that $c_\delta \subseteq E$, therefore the sequences $\langle P_\delta(\overline{G}, \overline{C}) : \delta \in S \rangle$ and $\langle P_\delta(\overline{G}', \overline{C}) : \delta \in S \rangle$ are equivalent modulo $\mathrm{id}(\overline{C})$. ∎

We remark at this point that the definition just made depends on the existence of a club guessing sequence! Strangely enough, we can prove the existence of club guessing sequences for regular uncountable cardinals λ for all such cardinals **except $\aleph_1$.**

Let us now quote the relevant theorems which assert the existence of club guessing sequences:

1.11 THEOREM: *If μ and λ are cardinals, $\mu^+ < \lambda$ and λ is regular, then there is a club guessing sequence $\langle c_\delta \colon \delta \in S \rangle$ such that the order type of each c_δ is $\geq \mu$.*

Proof: In [Sh-e, new VI§2]= [Sh-e, old III§7].

We proceed to show that Inv is preserved, in a way, under pure embeddings.

1.12 LEMMA: *Suppose that H and G are groups of cardinality λ and that $\overline{H}$ and $\overline{G}$ are λ-filtrations. Suppose that $\overline{C}$ is a club guessing sequence on $S \subseteq \lambda$. If $h \colon H \to G$ is a pure embedding, then for almost every $\delta \in S$, $P_\delta(\overline{H}, \overline{C}) \subseteq P_\delta(\overline{G}, \overline{C})$.*

Proof: Suppose for simplicity that $H \subseteq_{pr} G$, namely that the embedding is the identity function. The set $E_1 \stackrel{\text{def}}{=} \{\alpha \colon H \cap G_\alpha = H_\alpha\}$ is a club. Define for every natural number n a function $f_n(y)$ on G as follows:

$$f_n(y) = \begin{cases} \text{some } x \in \{x \colon x \in H \,\&\, (x+y) \in nG\} & \text{if } \{x \colon x \in H \,\&\, (x+y) \in nG\} \\ & \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

There is a club E_2 such that G_α is closed under f_n for all n for every $\alpha \in E_2$. $E = E_1 \cap E_2$ is a club.

1.13 CLAIM: *Suppose that $h \in H$ and that $\alpha \in E$. Then h has an n-congruent in G_α (in the sense of G) iff h has an n-congruent in H_α (in the sense of H).*

Proof: One direction is trivial. Suppose, then, that there is an n-congruent $g \in G_\alpha$. Let $h' = f_n(g)$. By the definition of f_n, $h' + g \in nG_\alpha$; also $h - g \in nG$. Therefore $h - h' \in nG$. As $H \subseteq_{pr} G$, $h - h' \in nH$, and therefore h' is an n-congruent of h in the sense of H. ∎

The proof of the Lemma follows now readily: For almost every $\delta \in S$ it is true $c_\delta \subseteq E$. Therefore for every such δ, every $h \in H$ and every n, h has an n-congruent in H_α iff h has an n-congruent in G_α. Therefore $\text{Inv}_{\overline{H}}(h, c_\delta) \subseteq \text{Inv}_{\overline{G}}(h, c_\delta)$. ∎

2. Constructing groups with prescribed INV

In this section we construct several groups with prescribed demands on their INV. These groups are used in the next section to show that in certain cardinals universal groups do not exist. The method in all constructions is attaching to a simply defined group points from a topological completion of the group.

a. CONSTRUCTIONS OF p-GROUPS.

2.1 THEOREM: *If λ is a regular uncountable cardinal, $\overline{C} = \langle c_\delta \colon \delta \in S \rangle$ is a club guessing sequence and $A_\delta \subseteq c_\delta$ is a given set of order type ω, then there is a separable p-group G of cardinality λ and λ-filtration $\overline{G}$ such that $A_\delta \in P_\delta(\overline{G}, \overline{C})$ for every $\delta \in S$.*

2.2 *Remark:* This implies by Lemma 1.12 that for every separable p-group G' of cardinality λ which purely extends G and a λ-filtration $\overline{G}'$, for almost every δ, $A_\delta \in P_\delta(\overline{G}', \overline{C})$.

Proof: For every n, let $B_n = \bigoplus_{\eta \in {}^n \lambda} A_\eta$ where A_η is a copy of Z_{p^n} with generator a_η. Let $G^0 = \bigoplus_n B_n$, and let G^1 be the torsion completion of G^0. G^1 may be identified with all sequences $(x_1, x_2, \ldots)$ where $x_n \in B_n$ and such that there is a (finite) bound to $\{o(x_n)\}_n$. For details see [Fuchs II,14–21]. The group we seek lies between G^0 and G^1, and is a pure subgroup of G^1.

Let us make a simple observation:

(1) If $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ belong to G^1 and $x - y \in p^n G^1$, then $x_i = y_i$ for all $i \leq n$.

Proof: Let $z_i = x_i - y_i$. $z_i \in B_i$. As $B_i \cap p^n G^1 = 0$ for $i \leq n$, we are done. ∎

For every $\delta \in S$ let $\langle \alpha_n^\delta \colon n \in \omega \rangle$ be the increasing enumeration of A_δ. Denote by η_n^δ the sequence $\langle \alpha_1^\delta, \ldots, \alpha_n^\delta \rangle$. Let $b_\delta^0 \in G^1$ be $(x_1^\delta, x_2^\delta, \ldots)$ where $x_n^\delta = p^{n-1} a_{\eta_n^\delta}$. So x_n is of order p and height $n - 1$. Consequently, b_δ^0 is of order p. Let us denote

$$\frac{x_k^\delta}{p^n} \overset{\text{def}}{=} \begin{cases} p^{k-n-1} a_{\eta_k^\delta} & \text{if } n < k \\ 0 & \text{otherwise} \end{cases}$$

and also let $\frac{0}{p^n} \overset{\text{def}}{=} 0$. Let $b_\delta^n = (\ldots, \frac{x_k^\delta}{p^n}, \ldots)$. Let G be the subgroup of G^1 generated by G^0 together with $\{b_\delta^n \colon \delta \in S, \, n < \omega\}$. Having defined G, let us specify a λ-filtration $\overline{G}$. For every $i < \lambda$ let G_i be $\langle \{a_\eta \colon \eta \in {}^{<\omega} i\} \cup \{b_\delta^n \colon \delta \in S, \, \delta < i, \, n < \omega\} \rangle$.

2.3 CLAIM: $\mathrm{Inv}_{\overline{G}}(b_\delta^0, c_\delta) = A_\delta$.

Proof: We should show that the set of indices i with the property that in G_{i+1} some congruent of b_δ^0 appears coincides with A_δ. Suppose first, then, that $i = \alpha_n$ for some n. $(x_1^\delta, \ldots, x_n^\delta, 0, 0, \ldots)$ is clearly a p^n-congruent of b_δ^δ,

as $b_\delta^0 - (x_1^\delta, \ldots, x_n^\delta) = p^n b_\delta^n$. Conversely, suppose that $i < \alpha_n$ and suppose to the contrary that there is some $y = (y_1, y_2, \ldots) \in G_i$ such that $b_\delta^0 - y \in P^n G$. Then $y_i = x_i$ for all $i \leq n$ by (1). But $y \in G_i$ implies that $y_n \in G_i$ — a contradiction to $i < \alpha_n$. ∎

b. CONSTRUCTIONS OF TORSION-FREE GROUPS. We start by constructing a torsion-free homogeneous group of a given type $\mathbf{t} = (\infty, \ldots, \infty, 0, \infty, \ldots)$. We recall that a characteristic $\chi(g)$ of an element $g \in G$ is the sequence $(k_1, k_2, \ldots)$ where k_l is the p-height of g for the l-th prime. A height can be ∞. A type $\mathbf{t}$ is an equivalence class of characteristics modulo the equivalence relation of having only a finite difference in a finite number of coordinates. A homogeneous group is a group in which all elements have the same type. We call a type $\mathbf{t}$ a p-type if $\mathbf{t} = (\infty, \ldots, \infty, 0, \infty, \ldots)$ where the only coordinate in which there is 0 is the number of p in the list of primes.

2.4 THEOREM: *For every uncountable and regular cardinal λ, a club guessing sequence $\overline{C} = \langle c_\delta \colon \delta \in S \rangle$ and given sets $A_\delta \in c_\delta$, each A_δ of order type ω, there is a homogeneous group G of cardinality λ with p-type $\mathbf{t}$ and a λ-filtration $\overline{G}$ such that for every $\delta \in S$, $A_\delta \in P_\delta(\overline{G}, \overline{C})$.*

2.5 *Remark:* This means that for every pure extension G' of G, for almost every $\delta \in S$, $A_\delta \in P_\delta(\overline{G'}, \overline{C})$.

Proof: This proof resembles the proof of Theorem 2.1. Let $G^0 = \bigoplus_\lambda Q_p$ (where Q_p is the group or rationals with denominators prime to p). We index the isomorphic copies of Q_p by $\eta \in {}^n\lambda$ and fix a_η, an element a_η of characteristic $(\infty, \ldots, \infty, 0, \infty, \ldots)$ in the η-th copy of Q_p. Let G^1 be the completion of G^0 in the p-adic topology. Let $\langle \eta^\delta(n) \colon n < \omega \rangle$ be the increasing enumeration of A_δ, and let $\eta_n^\delta = \langle \eta^\delta(0), \ldots, \eta^\delta(n-1) \rangle$. Let $b_{\delta,n} = \sum_k p^{k-n} a_{\eta_k}$. The rest is as in the proof of Theorem 2.1. ∎

3. The main results

a. THE UNIVERSALITY SPECTRUM OF TORSION GROUPS. There is a universal torsion group in λ iff there is a universal p-group in λ for every prime p. We therefore may focus on p-groups alone. There is a universal divisible p-group in λ, the group $\bigoplus_\lambda Z(p^\infty)$, therefore the first interesting question to ask in torsion groups is whether there is a universal **reduced** p-group. Here the answer is "no":

3.1 THEOREM: *If λ is an infinite cardinal (not necessarily regular, not necessarily uncountable), then there is no universal reduced p-group in λ.*

Proof: There are p-groups of cardinality λ of Ulm length σ for every ordinal $\sigma < \lambda^+$. As $u(A) \leq u(B)$ whenever $A \subseteq B$, and $u(A) < \lambda^+$, for every group of cardinality λ, no p-group of cardinality λ can be universal.   ∎

We put a further restriction on the class of p-groups, by demanding that the Ulm length of a group be at most ω.* We restrict ourselves then to the class of separable p-groups. On this class see [Fu] vol II, chapter XI.

b. UNIVERSAL SEPARABLE p-GROUPS.   We investigate the universality spectrum of the class of separable p-groups.

3.2 THEOREM: *If $\lambda = \lambda^{\aleph_0}$ then there is a purely universal separable p-group in λ.*

Proof: Let $B = \bigoplus B_n$ where $B_n = \bigoplus_\lambda Z_{p^n}$. The torsion completion of B, denoted by G, is of cardinality $|B|^{\aleph_0} = \lambda^{\aleph_0} = \lambda$ and is puely universal in λ. To see this let A be any separable p-group of cardinality λ, and let B_A be its basic subgroup. B_A is purely embeddable in B, and this gives rise to a pure embedding of A into G.   ∎

We see then, that for every n such that $\aleph_n \geq 2^{\aleph_0}$ there is a purely universal separable p-group in $\aleph_n$. As CH implies that in every $\aleph_n^{\aleph_0} = \aleph_n$ for all n, it follows by 3.2 that there is a purely universal separable p-group in every $\aleph_n$. It is not uncommon that CH decides questions in algebra. It is much less common, though, that a negation of CH or of GCH does the same. The following theorem uses a negation of GCH as one of its hypotheses.

3.3 THEOREM: *λ is regular and there is some μ such that $\mu^+ < \lambda < \mu^{\aleph_0}$, then there is no purely universal separable p-group in λ.*

Proof: By $\mu^+ < \lambda$ and Theorem 1.11, we may pick some club guessing sequence $\overline{C} = \langle c_\delta : \delta \in S \rangle$ where S is a stationary set of λ and otp $c_\delta \geq \mu$. Suppose G is a given separable p-group. We will show that G is not universal by presenting a separable p-group H of cardinality λ which is not embeddable in G. We choose a λ-filtration $\overline{G}$ of G and observe that $|P_\delta(\overline{G}, \overline{C})| \leq \lambda$ for every δ. As $\mu^{\aleph_0} > \lambda$,

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* One can make finer distictions here by considering the class of all p-groups of Ulm length which is bounded by an ordinal σ. But we do not do this here.

there is some $A_\delta \subseteq c_\delta$, of order type ω, which does not belong to $P_\delta(\overline{G}, \overline{C})$. By Theorem 2.1, there is a group H of cardinality λ such that for every embedding $\phi\colon H \to G$, for almost every δ, $A_\delta \in P_\delta(\overline{G}, \overline{C})$. This can hold only emptily, that is, if there are no such embeddings, because A_δ was chosen such that $A_\delta \notin P_\delta(\overline{G}, \overline{C})$. ∎

3.4 COROLLARY: *For $n \geq 2$, there is a purely universal separable p-group in $\aleph_n$ if, and only if, $2^{\aleph_0} \leq \aleph_n$.*

Proof: If $\aleph_n \geq 2^{\aleph_0}$ then $\aleph_n^{\aleph_0} = \aleph_n$ and by Theorem 3.2 there is a purely universal separable p-group in $\aleph_n$. Conversely, if $n \geq 2$ and $\aleph_n < 2^{\aleph_0}$, then by 3.3 there is no purely universal separable p-group in $\aleph_n$ ∎

c. THE UNIVERSALITY SPECTRUM OF TORSION-FREE GROUPS. We may restrict discussion in this Section to reduced torsion-free groups. We proceed to show first that in regular λ which satisfy $\lambda = \lambda^{\aleph_0}$ there is a universal reduced torsion-free group. The proof of the next theorem is an isolated point in the paper with respect to the technique, because it employs model theoretic notions (first order theory, elementary embedding and saturated model). These are available in every standard textbook on model theory, like [CK].

3.5 THEOREM: *If $\lambda = \lambda^{\aleph_0} \geq 2^{\aleph_0}$, then there is a universal reduced torsion-free group in λ.*

Proof: Let T be a complete first order theory of torsion free-groups. It is enough to find a reduced group G_T of cardinality λ such that $G_T \models T$ and for every $H \models T$, H is embedded in G_T; for if we have such a G for every T, the group $\bigoplus_T G_T$ is of cardinality λ (there are only $2^{\aleph_0}$ complete first order theories), and is evidently universal.

Let, then, G'_T be a saturated model of T of cardinality λ. Let D be its maximal division subgroup, and let $G_T \stackrel{\text{def}}{=} G'_T/D$. G_T is isomorphic to the direct summand of D, and is therefore torsion-free and reduced. Suppose that $H \models T$ is reduced (and, clearly, torsion-free). There is an elementary embedding $f\colon H \to G'_T$.

3.6 CLAIM: $\operatorname{Im} f \cap D = 0$

Proof: Suppose $0 \neq a \in H$ and $f(a) \in D$. As f is elementary, a is divisible in H by every integer n. As H is torsion-free, the set of all divisors of a generates a divisible subgroup of H, contrary to H being reduced.

We conclude, therefore, that $\hat{f}$ defined by $\hat{f}(a) = f(a) + D$ is an embedding of H into G_T. ∎

Next we show that below the continuum there is no purely-universal reduced torsion-free group. The reason for this is trivial: there are $2^{\aleph_0}$ types (over the empty set) in this class. Therefore we do not need the club guessing machinery, and gain an extra case — the case where $\lambda = \aleph_1$ — in comparison to Corollary 3.4.

3.7 THEOREM: *If $\lambda < 2^{\aleph_0}$ then there is no purely-universal reduced torsion-free group in cardinality λ. In fact, for every reduced torsion-free group G of cardinality λ there is a rank-1 group R which is not purely embeddable in G.*

Proof: As $\lambda < 2^{\aleph_0}$, there is a characteristic $(k_1, k_2, \ldots)$, with all k_i finite, which is not equal to $\chi_G(g)$ for every $g \in G$ (the definitions of **characteristic** and **type** are from [Fu], II, 85). Let R be a rank-1 group such that $\chi_R(1) = (k_1, k_2, \ldots)$. As pure embeddings preserve the characteristic, R is not purely embeddable in G. ∎

We look now at a class of torsion-free groups which do not have many types above the empty set. This is the class of homogeneous groups (a group is **homogeneous** if all non-zero elements in the group have the same type. See [Fu] II p.109). Here we are able again to prove that there is no purely universal group in the class in cardinality $\lambda < 2^{\aleph_0}$ if $\lambda > \aleph_1$. However, rather than using types over the empty set, we are using invariants.

3.8 THEOREM: *If λ is a regular cardinal, $\mu^+ < \lambda < \mu^{\aleph_0}$ for some μ, and $\mathbf{t}$ is a given p-type, then there is no purely-universal homogeneous torsion-free group of type $\mathbf{t}$ in λ. Even more: for every torsion-free group G of size λ there is a homogeneous group of size λ whose type is $\mathbf{t}$ which is not purely embedded in G.*

Proof: Let G be any torsion-free group of size λ, and fix some λ-filtration $\overline{G}$. Let $\overline{C}$ be a club guessing sequence, and for every $\delta \in S$ let A_δ be such that $A_\delta \notin P_\delta(\overline{G}, \overline{C})$. Such an A_δ exists, as $|P_\delta(\overline{G}, \overline{C})| \leq \lambda < 2^{\aleph_0}$, while there are $2^{\aleph_0}$ subsets of c_δ. By Theorem 2.4, there is a homogeneous group H with type $\mathbf{t}$ such that $A_\delta \in P_\delta(\overline{H}, \overline{C})$ for every $\delta \in S$. If there were a pure embedding $\phi: H \to G$, then by Theorem 1.12, for almost every $\delta \in S$, A_δ would be in $P_\delta(\overline{G}, \overline{C})$. But by the choice of A_δ this is impossible. ∎

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